

Varieties of Picard rank one as components of ample divisors

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Abstract

Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ and denote by \mathcal{L} an ample line bundle on \mathcal{V} . By imposing that the linear system $|\mathcal{L}|$ contains an element $A = A_1 + \dots + A_r, r \geq 1$, where all the A_i 's are distinct effective Cartier divisors with $\text{Pic}(A_i) = \mathbb{Z}$, we show that such a \mathcal{V} is as special as the components A_i of $A \in |\mathcal{L}|$. After making a list of some consequences about the positivity of the components A_i , we characterize pairs $(\mathcal{V}, \mathcal{L})$ as above when either $A_1 \cong \mathbb{P}^{n-1}$ and $\text{Pic}(A_j) = \mathbb{Z}$ for $j = 2, \dots, r$, or \mathcal{V} is smooth and each A_i is a variety of small degree with respect to $[H_i]_{A_i}$, where $[H_i]_{A_i}$ is the restriction to A_i of a suitable line bundle H_i on \mathcal{V} .¹

1 Introduction

Projective manifolds with an irreducible hyperplane section being a special variety have been studied since longtime (see, e.g., [2] and [6]), but the corresponding study for a reducible hyperplane section consisting of a simple normal crossing divisor whose components are special varieties started only recently by Chandler, Howard and Sommese [3]. Therefore, we continue here the study of varieties in terms of a hyperplane section A which is not irreducible, assuming that A is a union of distinct irreducible components A_1, \dots, A_r , with $r \geq 1$. More precisely, let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ endowed with an ample line bundle \mathcal{L} . Assume that

- (\diamond) $|\mathcal{L}|$ contains an element $A = A_1 + \dots + A_r, r \geq 1$, where all the components A_i are distinct and effective Cartier divisors with $\text{Pic}(A_i)$ of rank one.

Let us observe here that this assumption is a natural generalization of the classical hypothesis $\text{Pic}(A') \cong \mathbb{Z}$ on a hyperplane section A' of \mathcal{V} . Furthermore, if (\diamond) holds then every component A_i of $A \in |\mathcal{L}|$ does not admit a non-trivial morphism onto a variety W_i with $0 < \dim W_i < n - 1$ and in general, also for the smooth case, the main known results on reducible hyperplane sections

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(see, e.g., [3]) can not be applied on any of the A_i 's. However, we can show that if a reducible subvariety A as in (\diamond) is contained in \mathcal{V} as an ample divisor, then it imposes severe restrictions to \mathcal{V} , that is, the topological and geometric structures of \mathcal{V} are very closely related to those of each component of A .

So, in §3 we prove first the following

Theorem 1.1. *Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ and let \mathcal{L} be an ample line bundle on \mathcal{V} . Assume that (\diamond) holds. Then all the A_i 's are nef and big Cartier divisors on \mathcal{V} and for any $i = 1, \dots, r$ there exist proper birational morphisms $f_i : \mathcal{V} \rightarrow \mathcal{V}_i$ from \mathcal{V} to a projective normal variety \mathcal{V}_i given by the map associated to $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$ for some $m_i \gg 0$. Furthermore, each map f_i contracts at most a finite number of curves on \mathcal{V} and it is an isomorphism in a neighborhood of A_i such that $f_i(A_i)$ is an effective ample divisor on \mathcal{V}_i .*

The above result shows that assumption (\diamond) implies that all the components A_i of A are either ample, or at worst big and 1-ample in the sense of [15, (1.3)]. By imposing some restrictions on the singularities of \mathcal{V} , we are able to deduce that all the A_i 's are in fact ample Cartier divisors on \mathcal{V} .

Theorem 1.2. *Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen-Macaulay singularities. Let \mathcal{L} be an ample line bundle on \mathcal{V} and assume that (\diamond) holds. Furthermore, suppose that $\mathcal{V} - F$ is a locally complete intersection for some finite, possibly empty, set $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$ with $\dim \text{Irr}(\mathcal{V}) \leq 0$, where $\text{Irr}(\mathcal{V})$ is the set of irrational singularities of \mathcal{V} . Then all the A_i 's are ample Cartier divisors on \mathcal{V} and the maps $f_i : \mathcal{V} \rightarrow \mathcal{V}_i$ of Theorem 1.1 are all isomorphisms.*

Furthermore, under some additional hypotheses on \mathcal{V} and on some A_i , we can finally obtain that $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ for an ample line bundle Λ on \mathcal{V} (see Corollary 3.1).

All of these results allow us to list in §4 some consequences about the positivity of the A_i 's and to obtain similar results as in [1, Theorem 1] (see also [14, Prop.VI]) for the case of reducible ample divisors on \mathcal{V} (see Propositions 5.2 and 5.3).

We would like to note that the above results make use of weak hypotheses on \mathcal{V} and on each A_i , and that (\diamond) seems optimal a priori for Theorems 1.1 and 1.2, since easy examples show that these results do not hold assuming that $\text{Pic}(A_i) \neq \mathbb{Z}\langle \mathcal{H}_i \rangle$ for some $i = 1, \dots, r$, also when $r = 2$ and \mathcal{V} is smooth (Remark 3.2). Moreover, the techniques we employ in §3 leave out of account any special polarization on each A_i by a (very) ample line bundle on \mathcal{V} and they allow us to assume that \mathcal{L} is simply ample and not necessarily ample and spanned or very ample on \mathcal{V} .

Finally, as a by-product of §4 and some results obtained by many other authors about smooth complex projective variety X containing ample divisors of special type (e.g., [1], [2], [6], [7], [9], [10], [11], [14]), in §5.1 we obtain similar results as in [1, Theorem 1] and [14, Prop.VI] (see Propositions 5.2, 5.3), and

in §5.2 we classify smooth polarized pairs (X, L) which admit an ample divisor $A \in |L|$ such that $A = A_1 + \dots + A_r$, $r \geq 1$, and all the components A_i have small degree with respect to suitable line bundles H_i on X for every $i = 1, \dots, r$.

Proposition 1.3. *Let L be an ample line bundle on a smooth complex projective variety X of dimension n with $n \geq 5$. Assume that there is a divisor $A = A_1 + \dots + A_r \in |L|$, $r \geq 1$, where each A_i is an irreducible and reduced normal Gorenstein projective variety with $\dim \text{Irr}(A_i) \leq 0$, where $\text{Irr}(A_i)$ is the set of irrational singularities of A_i . Suppose that for any $k = 1, \dots, r$ there exist ample and spanned line bundles H_k on X such that $[H_k]_{A_k}$ is very and $[H_k]_{A_k}^{n-1} \leq 4$. Then one of the following possibilities holds:*

1. $r \geq 1$, $H_1 = \dots = H_r = H$ and (X, H) is one of the following pairs:

- (a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ and $A_i \in |\mathcal{O}_{\mathbb{P}^n}(a_i)|$ with $1 \leq a_i \leq 4$ for every $i = 1, \dots, r$;
- (b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ and $A_i \in |\mathcal{O}_{\mathbb{Q}^n}(a_i)|$ with $a_i = 1, 2$ for every $i = 1, \dots, r$;
- (c) $X \subset \mathbb{P}^{n+1}$ is a hypersurface of degree 3 or 4, and $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+1}}(1)_X|$ for every $i = 1, \dots, r$;
- (d) $X = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$ is a complete intersection of two quadric hypersurfaces $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$ for $i = 1, 2$, and $A_i = H \in |\mathcal{O}_{\mathbb{P}^{n+2}}(1)_X|$ for every $i = 1, \dots, r$;
- (e) $\pi : X \rightarrow \mathbb{P}^n$ is a double cover of \mathbb{P}^n with branch locus $\Delta \in |\mathcal{O}_{\mathbb{P}^n}(2b)|$, $b = 1, 2$, $H \in |\pi^* \mathcal{O}_{\mathbb{P}^n}(1)|$ and $A_i \in |\pi^* \mathcal{O}_{\mathbb{P}^n}(a_i)|$, $a_i = 1, 2$, for every $i = 1, \dots, r$;
- (f) $\pi : X \rightarrow \mathbb{Q}^n$ is a double cover of a quadric hypersurface $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$ with branch locus $\Delta \in |\mathcal{O}_{\mathbb{Q}^n}(2b')|$, $b' = 1, 2$, and $A_i = H \in |\pi^* \mathcal{O}_{\mathbb{Q}^n}(1)|$ for every $i = 1, \dots, r$;
- (g) $\pi : X \rightarrow \mathbb{P}^n$ is a d -cover of \mathbb{P}^n with $d = 3, 4$, and $A_i = H \in |\pi^* \mathcal{O}_{\mathbb{P}^n}(1)|$ for any $i = 1, \dots, r$;

2. $r \geq 2$, $X \cong \mathbb{P}^1 \times \mathbb{P}^4$ and after renaming $(A_1, [H_1]_{A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$, with $A_1 \in |\mathcal{O}_X(0, 1)|$ and $H_1 \in |\mathcal{O}_X(1, 1)|$, $(A_2, [H_2]_{A_2}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$ with $A_2 \in |\mathcal{O}_X(1, 0)|$ and $H_2 \in |\mathcal{O}_X(t, 1)|$ for some integer $t \geq 1$, and the remaining polarized pairs $(A_k, [H_k]_{A_k})$ are of these two types.

2 Notation

Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ endowed with an ample line bundle \mathcal{L} . All notation and terminology used here are standard in algebraic geometry. We adopt the additive notation for line bundles and the numerical equivalence is denoted by \equiv , while the linear equivalence by \simeq . The pull-back $\iota^* \mathcal{F}$ of a line bundle \mathcal{F} on \mathcal{V} by an embedding $\iota : Y \hookrightarrow \mathcal{V}$ is sometimes denoted by \mathcal{F}_Y . We denote by $K_{\mathcal{V}}$ the canonical bundle of \mathcal{V} . For such a polarized variety $(\mathcal{V}, \mathcal{L})$, we will use the adjunction theoretic terminology of [2] and we say that \mathcal{V} is an n -fold (denoted by X) when \mathcal{V} is smooth.

3 Proof of Theorems 1.1 and 1.2

First of all, let us show that (\diamond) implies that all the components of A are actually nef and big Cartier divisors on \mathcal{V} , obtaining the following

Proof of Theorem 1.1. Suppose that $A_1 \cap A_2 := W$ is nonempty. Define integers a_{ij} putting $\mathcal{O}_{\mathcal{V}}(A_i)_{A_j} := \mathcal{O}_{A_j}(a_{ij}\mathcal{H}_j)$, where \mathcal{H}_j is the ample generator of $\text{Pic}(A_j) \pmod{\text{torsion}}$. Then

$$\mathcal{O}_W(a_{11}\mathcal{H}_1) = \mathcal{O}_W(A_1) = \mathcal{O}_W(a_{12}\mathcal{H}_2)$$

is an ample line bundle on W . Hence a_{11} is positive. Since A is a connected divisor on \mathcal{V} (see [8, III 7.9]), for every $i = 1, \dots, r$ there exists a component A_j of $A \in |\mathcal{L}|$ with $j \neq i$ such that $A_i \cap A_j \neq \emptyset$. This shows that $\mathcal{O}_{\mathcal{V}}(A_i)_{A_i}$ is ample for any $i = 1, \dots, r$, i.e. A_i is nef and big. From [7, III 4.2] (see also [2, (2.6.5)] or [12, (1.2.30)]), it follows that there exists a proper birational morphism $f_i : \mathcal{V} \rightarrow \mathcal{V}_i$ from \mathcal{V} to a projective normal variety \mathcal{V}_i given by the map associated to $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$ for some $m_i \gg 0$, which is an isomorphism in a neighborhood of A_i and such that $f_i(A_i)$ is an effective ample divisor on \mathcal{V}_i .

Claim. The morphism f_i contracts at most a finite number of curves on \mathcal{V} .

By [2, (2.5.5)] note that $A_i \simeq H_i + D_i$ for any $i = 1, \dots, r$, where H_i is \mathbb{Q} -ample and D_i is \mathbb{Q} -effective. This shows that $A_i \cdot A_j$ is nonzero and so $A_i \cap A_j$ can not be empty. Let $W_{ij} := A_i \cap A_j$ and note that W_{ij} is an ample divisor both in A_i and in A_j . Let Z be an irreducible variety of dimension greater than or equal to two. Since \mathcal{L} is ample, we have that $Z \cap A_k \neq \emptyset$ for some $k = 1, \dots, r$, and then $W_{ik} \cap Z \neq \emptyset$ since $\dim Z \cap A_k \geq 1$. Put $R := Z \cap A_i$. Thus R is nonempty, $\dim R \geq 1$ and $\mathcal{O}(R)_R$ is ample. By [7, III 4.2] (or [12, (1.2.30)]), the linear system $|\mathcal{O}_{\mathcal{V}}(m_i A_i)|$ restricted to Z can not contract Z to a lower dimensional variety.

Proof of Theorem 1.2. Put $\mathcal{O}_{\mathcal{V}}(A_i)_{A_j} = \mathcal{O}_{A_j}(a_{ij}\mathcal{H}_j)$ and $l_t := \mathcal{O}_{A_t}(\mathcal{H}_t)^{n-1} > 0$ for any $i, j, t \in \{1, \dots, r\}$. From Theorem 1.1 we know that $a_{ii} > 0$ for every $i = 1, \dots, r$. Moreover, since A is a connected divisor on X (see [8, III 7.9]), for every $j = 1, \dots, r$ there exists a component A_k of $A \in |L|$ with $k \neq j$ such that $A_j \cap A_k \neq \emptyset$. So we get the following expressions

$$\begin{aligned} A_k^s A_j^{n-s} &= \mathcal{O}_{A_k}(a_{kk}\mathcal{H}_k)^{s-1} \mathcal{O}_{A_k}(a_{jk}\mathcal{H}_k)^{n-s} = a_{kk}^{s-1} a_{jk}^{n-s} l_k \\ A_k^s A_j^{n-s} &= \mathcal{O}_{A_j}(a_{kj}\mathcal{H}_j)^s \mathcal{O}_{A_j}(a_{jj}\mathcal{H}_j)^{n-s-1} = a_{kj}^s a_{jj}^{n-s-1} l_j, \end{aligned} \quad (1)$$

with $1 \leq s \leq n-1$. Note that $a_{jj} \neq 0$, $a_{kk} \neq 0$, $a_{kj} > 0$ and $a_{jk} > 0$. Moreover, from the equations (1) with $s = 1, 2$ we deduce that

$$a_{jk}(a_{kj}^2 a_{jj}^{n-3} l_j) = a_{jk}(a_{kk} a_{jk}^{n-2} l_k) = a_{kk}(a_{jk}^{n-1} l_k) = a_{kk}(a_{kj} a_{jj}^{n-2} l_j),$$

that is,

$$a_{jk} a_{kj} = a_{jj} a_{kk}. \quad (2)$$

So we get

$$\begin{aligned}
A_s^2 \mathcal{L}^{n-2} &= \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} A_1^{h_1} \dots A_s^{h_s+2} \dots A_r^{h_r} \\
&= \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} a_{1s}^{h_1} \dots a_{ss}^{h_s+1} \dots a_{rs}^{h_r} l_s \\
&= a_{ss}^{n-1} l_s + [\text{non-negative terms}] > 0,
\end{aligned}$$

for every $s = 1, \dots, r$. Furthermore, for $j \neq k$ we have also the following equations

$$A_j^2 \mathcal{L}^{n-2} = [A_j]_{A_j} \mathcal{L}_{A_j}^{n-2} = \sum_{k_1+\dots+k_r=n-2} \frac{(n-2)!}{k_1! \dots k_r!} a_{1j}^{k_1} \dots a_{jj}^{k_j+1} \dots a_{rj}^{k_r} l_j \quad (3)$$

$$A_k^2 \mathcal{L}^{n-2} = [A_k]_{A_k} \mathcal{L}_{A_k}^{n-2} = \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} a_{1k}^{h_1} \dots a_{kk}^{h_k+1} \dots a_{rk}^{h_r} l_k \quad (4)$$

$$A_j A_k \mathcal{L}^{n-2} = [A_k]_{A_j} \mathcal{L}_{A_j}^{n-2} = \sum_{k_1+\dots+k_r=n-2} \frac{(n-2)!}{k_1! \dots k_r!} a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{kj}^{k_k+1} \dots a_{rj}^{k_r} l_j \quad (5)$$

$$A_j A_k \mathcal{L}^{n-2} = [A_j]_{A_k} \mathcal{L}_{A_k}^{n-2} = \sum_{h_1+\dots+h_r=n-2} \frac{(n-2)!}{h_1! \dots h_r!} a_{1k}^{h_1} \dots a_{jk}^{h_j+1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r} l_k. \quad (6)$$

Since by (2) we get

$$\begin{aligned}
&(a_{1j}^{k_1} \dots a_{jj}^{k_j+1} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{kk}^{h_k+1} \dots a_{rk}^{h_r}) = \\
&= a_{jj} a_{kk} (a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r}) = \\
&= a_{jk} a_{kj} (a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r}) = \\
&= (a_{1j}^{k_1} \dots a_{jj}^{k_j} \dots a_{kj}^{k_k+1} \dots a_{rj}^{k_r})(a_{1k}^{h_1} \dots a_{jk}^{h_j+1} \dots a_{kk}^{h_k} \dots a_{rk}^{h_r}),
\end{aligned}$$

from (3), (4), (5) and (6), we obtain that

$$(A_k A_j \mathcal{L}^{n-2})^2 = (A_k^2 \mathcal{L}^{n-2})(A_j^2 \mathcal{L}^{n-2}) > 0$$

for every k and j such that $A_j \cap A_k \neq \emptyset$.

Thus, by [2, (2.5.4)] we see that there exists a rational number λ_{jk} such that A_j is numerically equivalent to $\lambda_{jk} A_k$. Since from (2) it follows that A_k meets all the other components of A , by an inductive argument we get

$$\mathcal{L} \simeq A_1 + \dots + A_r \equiv (\lambda_{1k} + \dots + \widehat{\lambda_{kk}} + \dots + \lambda_{rk} + 1) A_k = \mu_k A_k, \quad (7)$$

where $\mu_k := \lambda_{1k} + \dots + \widehat{\lambda_{kk}} + \dots + \lambda_{rk} + 1$ and the symbol $\widehat{}$ denotes suppression.

Moreover, since

$$0 < \mathcal{L}_{A_j}^{n-1} = A_j \mathcal{L}^{n-1} = \lambda_{jk} A_k \mathcal{L}^{n-1} = \lambda_{jk} \mathcal{L}_{A_k}^{n-1},$$

we have that $\lambda_{jk} > 0$ for every $j \neq k$ and from (7) it follows that $\mu_k \geq 1$, i.e. A_k is an ample Cartier divisor on \mathcal{V} for any $k = 1, \dots, r$. By combining this with Theorem 1.1, we obtain that every f_i is an isomorphism.

Corollary 3.1. *Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen-Macaulay singularities. Let \mathcal{L} be an ample line bundle on \mathcal{V} and assume that (\diamond) holds. Furthermore, suppose that one of the following conditions holds:*

- (a) \mathcal{V} is a locally complete intersection;
- (b) $n = 3$, $\mathcal{V} - A_k$ and $\mathcal{V} - F$ are locally complete intersections for some $k = 1, \dots, r$ and some finite set $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$.

Then $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$, where Λ is an ample line bundle on \mathcal{V} . In particular, all the A_i 's are ample Cartier divisors on \mathcal{V} and the maps $f_i : \mathcal{V} \rightarrow \mathcal{V}_i$ of Theorem 1.1 are all isomorphisms.

Proof. In both cases (a) and (b), we simply use Theorem 1.2 and [2, (2.3.4)]. \square

Remark 3.2. Let $\mathcal{V} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-d))$ with $0 < d < n$ and denote by $\pi : \mathcal{V} \rightarrow \mathbb{P}^{n-1}$ the projection map. Note that there exists a smooth divisor E on \mathcal{V} which is a section of π such that $E \cong \mathbb{P}^{n-1}$ and $E_E \in |\mathcal{O}_{\mathbb{P}^{n-1}}(-d)|$. Put $\mathcal{L} := E + \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)$. Then, for a suitable integer $a \gg 0$, we have that \mathcal{L} is ample on \mathcal{V} and that there exists a divisor $A \in |\mathcal{L}|$ such that $A = A_1 + A_2$, where $A_1 = E$ and $A_2 \in |\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)|$ is a smooth divisor on \mathcal{V} . This example shows that if $r \geq 2$ then the above results can not be improved by assuming in (\diamond) that $\text{Pic}(A_i) \neq \mathbb{Z}$ for some $i = 1, \dots, r$, also when $r = 2$ and \mathcal{V} is a Fano n -fold with $n \geq 3$.

4 Some immediate consequences

Let us collect here some results due to the different positivity of the components A_i of $A \in |\mathcal{L}|$ under the hypothesis (\diamond) .

4.0.1 Nefness and bigness of the A_i 's

From Theorem 1.1, we obtain the following

Corollary 4.1. *Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ and let \mathcal{L} be an ample line bundle on \mathcal{V} . Assume that (\diamond) holds. Set $D = \sum A_{i_h}$, where all the $i_h \in \{1, \dots, r\}$ are not necessarily distinct indexes. Moreover, let $\text{Irr}(\mathcal{V})$ be the set of irrational singularities of \mathcal{V} . Then we have the following properties:*

- (Vanishing type Theorems).

(1) Let $\varphi : \mathcal{V} \rightarrow Y$ be a morphism from \mathcal{V} to a projective variety Y . Then

$$\varphi_{(i)}(K_{\mathcal{V}} + D) = 0 \quad \text{for } i \geq \max_{y \in \varphi(\text{Irr}(\mathcal{V}))} \dim(\varphi^{-1}(y) \cap \text{Irr}(\mathcal{V})) + 1,$$

where $\varphi_{(i)}(\mathcal{T})$ is the i -th higher derived functor of the direct image $\varphi_*(\mathcal{T})$ of a sheaf \mathcal{T} on \mathcal{V} ;

(2) $H^1(\mathcal{V}, -D) = 0$;

(3) assuming that $\text{Irr}(\mathcal{V})$ is finite and nonempty, we have that

$$\dim H^0(\mathcal{V}, K_{\mathcal{V}} + D) \geq \#(\text{Irr}(\mathcal{V})) > 0,$$

where $\#(\text{Irr}(\mathcal{V}))$ is the number of points in $\text{Irr}(\mathcal{V})$; in particular, if $\mathcal{A} \in |D|$ lies in the set of Cohen-Macaulay points of \mathcal{V} , then

$$\dim H^0(\mathcal{V}, K_{\mathcal{V}}) + \dim H^0(\mathcal{A}, K_{\mathcal{A}}) \geq \#(\text{Irr}(\mathcal{V})) > 0.$$

- (Lefschetz type Theorems).

(4) Let $\mathcal{A} \in |D|$ be a divisor such that $\mathcal{V} - \mathcal{A}$ is a local complete intersection. Then under the restriction map it follows that

$$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j \leq n - 3,$$

and

$$H^j(\mathcal{V}, \mathbb{Z}) \rightarrow H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j = n - 2,$$

is injective with torsion free cokernel; moreover, we have that $\text{Pic}(\mathcal{V}) \cong \text{Pic}(\mathcal{A})$ for $n \geq 5$, and the restriction mapping $\text{Pic}(\mathcal{V}) \rightarrow \text{Pic}(\mathcal{A})$ is injective with torsion free cokernel for $n = 4$; in particular, if $\mathcal{V} - \mathcal{A}_i$ is a local complete intersection for some $i = 1, \dots, r$, then $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ for $n \geq 4$ and some ample line bundle Λ on \mathcal{V} ;

- (The Albanese mapping).

(5) Assume that $\mathcal{A} \in |D|$ is normal. Moreover, suppose that \mathcal{A} and \mathcal{V} have at worst rational singularities. Then the map $\text{Alb}(\mathcal{A}) \rightarrow \text{Alb}(\mathcal{V})$ induced by inclusion is an isomorphism.

- (Hodge Index type Theorems).

(6) For $n_{i_1} + \dots + n_{i_k} = n - 1$ and $n_{i_1} \geq 1$, we have

$$(A_{i_0} \cdot A_{i_1}^{n_{i_1}} \dots A_{i_k}^{n_{i_k}})^2 \geq (A_{i_0}^2 \cdot A_{i_1}^{n_{i_1}-1} \dots A_{i_k}^{n_{i_k}})(A_{i_1}^{n_{i_1}+1} \dots A_{i_k}^{n_{i_k}}),$$

and for $n_{i_1} + \dots + n_{i_k} = n$, we get also

$$(A_{i_1}^{n_{i_1}} \dots A_{i_k}^{n_{i_k}})^n \geq (A_{i_1}^n)^{n_{i_1}} \dots (A_{i_k}^n)^{n_{i_k}} > 0,$$

where $i_h \in \{1, \dots, r\}$;

- (7) we have the following inequality: $(A_i^{n-1} \cdot A_j)(A_i \cdot A_j^{n-1}) \geq A_i^n A_j^n > 0$;
(8) for $t \geq 1$ and any nef and big line bundle H_i on \mathcal{V} with $1 \leq i \leq t$, it follows that $\mathcal{O}_{\mathcal{V}}(A_j)^{n-t} \cdot \prod_{i=1}^t H_i > 0$, where $j \in \{1, \dots, r\}$;
(9) let H be a line bundle such that $\dim H^0(\mathcal{V}, NH) \geq 2$ for some $N \geq 1$. Then $H \cdot \mathcal{O}_{\mathcal{V}}(A_{i_1}) \cdots \mathcal{O}_{\mathcal{V}}(A_{i_{n-1}}) > 0$, where $i_h \in \{1, \dots, r\}$.

Proof. First of all, from Theorem 1.1 we deduce that the effective divisor $D = \sum A_{i_h}$ is nef and big on \mathcal{V} . Thus (1), (2) and (3) of the statement follow from [2, (2.2.5), (2.2.7), (2.2.8)]. Finally, cases (4) to (9) follow from [2, (2.3.3), (2.3.4), (2.4.4), (2.5.1), (2.5.3), (2.5.2), (2.5.8) and (2.5.9)]. \square

By applying Theorem 1.1 also to the zero locus of special sections of k -ample vector bundles on an n -fold X for $k \geq 0$ (see [15, §1] and [2, §2.1]), we obtain the following

Corollary 4.2. (Lefschetz-Sommese type Theorem) *Let \mathcal{E} be a k -ample vector bundle of rank $r \geq 1$ on an n -fold X with $n \geq 4$. Assume that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is an integral normal complex variety such that $\dim Z \geq 3$. Suppose that there exists a divisor $A = A_1 + \dots + A_s$, $s \geq 1$, on Z which satisfies (\diamond) . If $Z - A_i$ is a local complete intersection for some $i = 1, \dots, s$, then*

$$H^i(X, \mathbb{Z}) \cong H^i(A_i, \mathbb{Z}) \quad \text{for } i \leq \min\{\dim Z - 3, n - r - k - 1\},$$

and the restriction maps $H^i(X, \mathbb{Z}) \rightarrow H^i(A_i, \mathbb{Z})$ are injective with torsion free cokernel for $i = \min\{\dim Z - 2, n - r - k\}$.

Proof. From Theorem 1.1, we know that each A_k is at worst 1-ample on Z . Thus by Corollary 4.1 (4), we have that

$$H^j(Z, \mathbb{Z}) \cong H^j(A_i, \mathbb{Z}) \quad \text{for } j \leq \dim Z - 3,$$

and the restriction maps $H^j(Z, \mathbb{Z}) \rightarrow H^j(A_i, \mathbb{Z})$ are injective with torsion free cokernel for $j = \dim Z - 2$. Moreover, from the Lefschetz-Sommese's Theorem for k -ample vector bundles on an n -fold X (see [15, (1.16)] and [13, (7.1.1), (7.1.9)]), we deduce that

$$H^j(Z, \mathbb{Z}) \cong H^j(X, \mathbb{Z}) \quad \text{for } j \leq n - r - k - 1,$$

and that the restriction maps $H^j(X, \mathbb{Z}) \rightarrow H^j(Z, \mathbb{Z})$ are injective with torsion free cokernel for $j = n - r - k$. \square

4.0.2 Ampleness of the A_i 's

Under the same assumption of Theorem 1.2, we first deduce the following

Corollary 4.3. *Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ with at worst Cohen-Macaulay singularities. Let \mathcal{L} be an ample line bundle on \mathcal{V} and assume that (\diamond) holds. Moreover, suppose that $\mathcal{V} - F$ is*

a local complete intersection for some finite, possibly empty, set $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$ with $\dim \text{Irr}(\mathcal{V}) \leq 0$, where $\text{Irr}(\mathcal{V})$ is the set of irrational singularities of \mathcal{V} . Set $D = \sum A_{i_h}$, where all the $i_h \in \{1, \dots, r\}$ are not necessarily distinct indexes. Then we have the following properties:

- (I) (Fujita's Vanishing type Theorem). Given any coherent sheaf \mathcal{F} on \mathcal{V} , there exists an integer $m(\mathcal{F}, D)$ such that

$$H^i(\mathcal{V}, \mathcal{F} \otimes \mathcal{O}_{\mathcal{V}}(mD + N)) = 0 \quad \text{for } i > 0, m \geq m(\mathcal{F}, D),$$

where N is any nef divisor on \mathcal{V} ;

- (II) $\dim H^0(\mathcal{V}, D) \leq D^n + n$, with equality if and only if \mathcal{V} is one of the following: (a) \mathbb{P}^n ; (b) a quadric hypersurface $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$; (c) a \mathbb{P}^{n-1} -bundle over \mathbb{P}^1 ; (d) a generalized cone over a smooth submanifold $V \subset \mathcal{V}$ as in (a), (b), (c);

- (III) (Lefschetz type Theorems). Let $\mathcal{A} \in |D|$ be a divisor such that $\mathcal{V} - \mathcal{A}$ is a local complete intersection. Then under the restriction map it follows that

$$H^j(\mathcal{V}, \mathbb{Z}) \cong H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j \leq n - 2,$$

and

$$H^j(\mathcal{V}, \mathbb{Z}) \rightarrow H^j(\mathcal{A}, \mathbb{Z}) \quad \text{for } j = n - 1,$$

is injective with torsion free cokernel; moreover, we have that $\text{Pic}(\mathcal{V}) \cong \text{Pic}(\mathcal{A})$ for $n \geq 4$, and the restriction mapping $\text{Pic}(\mathcal{V}) \rightarrow \text{Pic}(\mathcal{A})$ is injective with torsion free cokernel for $n = 3$; in particular, if $\mathcal{V} - A_i$ is a local complete intersection for some $i = 1, \dots, r$, then $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ for $n \geq 3$ and some ample line bundle Λ on \mathcal{V} .

Proof. By Nakai-Moishezon-Kleiman criterion and Theorem 1.2, we see that $D = \sum A_{i_h}$ is ample on \mathcal{V} . Thus case (I) of the statement follows from [5, §1, Th.1] (see also [12, (1.4.35)]). Finally, we obtain case (II) by [6, I (4.2), (5.10), (5.15)], while (III) follows from [2, (2.3.3)] and [2, (2.3.4)] respectively. \square

Finally, let us deduce also the following result for ample vector bundles on a smooth variety.

Corollary 4.4. *Let \mathcal{E} be an ample vector bundle of rank $r \geq 1$ on an n -fold X with $n \geq 4$. Assume that there exists a section $s \in \Gamma(\mathcal{E})$ whose zero locus $Z = (s)_0$ is a smooth submanifold of X . Suppose that there exists a divisor $A = A_1 + \dots + A_s$, $s \geq 1$, on Z which satisfies $\langle \diamond \rangle$. If $r < n - 2$ then both $\text{Pic}(X)$ and $\text{Pic}(Z)$ have rank one.*

Proof. It follows easily from Corollary 4.3 (III) and [15, (1.16)], or [13, (7.1.5)(ii)]. \square

5 Some applications

Here are two applications.

5.1 All the A_i 's are Fano varieties of Picard rank one

First of all, let us prove the following

Lemma 5.1. *Let $\mathcal{L}, \mathcal{V}, A_i, \mathcal{V}_i, f_i$ be as in Theorem 1.1. If \mathcal{V}_k is \mathbb{Q} -factorial for some $k = 1, \dots, r$, then A_k is ample on \mathcal{V} and $f_k : \mathcal{V} \rightarrow \mathcal{V}_k$ is in fact an isomorphism.*

Proof. Take any line bundle \mathcal{D} on \mathcal{V} and consider the following commutative diagram $(*)$:

$$\begin{array}{ccc} U & \xrightarrow{f_k|_U} & U' \\ j \downarrow & & \downarrow j_k \\ \mathcal{V} & \xrightarrow{f_k} & \mathcal{V}_k \end{array}$$

where $j : U \rightarrow \mathcal{V}$ and $j_k : U' \rightarrow \mathcal{V}_k$ are the inclusion maps and $f_k|_U : U \rightarrow U'$ is the isomorphism induced by $f_k : \mathcal{V} \rightarrow \mathcal{V}_k$. Since \mathcal{V}_k is \mathbb{Q} -factorial, we see that $f_{k*}(N\mathcal{D}) = \mathcal{L}'$ is a line bundle on \mathcal{V}_k for some positive integer N . Write $\mathcal{L}' = \sum_h a_h \mathcal{L}'_h$, where \mathcal{L}'_h are the generators of $\text{Pic}(\mathcal{V}_k)$. Then by $(*)$ we get

$$\begin{aligned} N\mathcal{D}|_U &= j^*(N\mathcal{D}) = f_k|_{U*} j^*(N\mathcal{D}) = j_k^* f_{k*}(N\mathcal{D}) = \sum_h a_h j_k^* \mathcal{L}'_h = \\ &= \sum_h a_h (f_k|_U)^* j_k^* \mathcal{L}'_h = \sum_h a_h j^* f_k^* \mathcal{L}'_h = j^* \left(\sum_h a_h f_k^* \mathcal{L}'_h \right) = \left(\sum_h a_h f_k^* \mathcal{L}'_h \right)|_U. \end{aligned}$$

By Hartogs' Lemma (see, e.g., [4, (11.4)]), this gives $N\mathcal{D} = \sum_h a_h f_k^*(\mathcal{L}'_h)$, i.e. $\mathcal{D} = \sum_h \frac{a_h}{N} f_k^*(\mathcal{L}'_h)$. Therefore, if A_k is not ample, then we deduce that there exists an irreducible curve $\Gamma \subset \mathcal{V}$ such that $m_k A_k \cdot \Gamma = 0$ for any positive integer m_k , i.e. the map f_k contracts the curve Γ . Hence $f_k^*(\mathcal{L}'_h) \cdot \Gamma = 0$ for any h , i.e. $\mathcal{D} \cdot \Gamma = 0$, but this leads to a contradiction by taking $\mathcal{D} = \mathcal{L}$. \square

Similar results as in [1, Theorem 1] (see also [14, Prop.VI]) for the case of reducible ample divisors on \mathcal{V} can be now proved.

Proposition 5.2. *Let \mathcal{V} be an integral normal complex projective variety of dimension $n \geq 3$ and let \mathcal{L} be an ample line bundle on \mathcal{V} . Assume that (\diamond) holds. If, up to renaming, $A_1 \cong \mathbb{P}^{n-1}$, then \mathcal{V} is the cone $\mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ on $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ with $A_1 = v_s(\mathbb{P}^{n-1})$ and $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ for a suitable integer $s > 0$, where v_s is the s^{th} Veronese embedding of \mathbb{P}^{n-1} .*

Proof. Since (\diamond) holds and $A_1 \cong \mathbb{P}^{n-1}$, by Theorem 1.1 we have that $f_1(A_1) \cong \mathbb{P}^{n-1}$ is ample on \mathcal{V}_1 . By [1, Theorem 1] we see that \mathcal{V}_1 is the cone $\mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$ over $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$, where s is a positive integer such that $\mathcal{N}_{f_1(A_1)/\mathcal{V}_1} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$.

Since \mathcal{V}_1 is \mathbb{Q} -factorial and $\text{Pic}(\mathcal{V}_1) = \mathbb{Z}$, by Lemma 5.1 we deduce that f_1 is an isomorphism and $\text{Pic}(\mathcal{V}) = \mathbb{Z}$. Therefore, A_1 is an ample divisor on

\mathcal{V} and by applying now [1, Theorem 1] to the pair (\mathcal{V}, A_1) , we get that $\mathcal{V} \cong \mathcal{C}(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(s))$, $A_1 \cong v_s(\mathbb{P}^{n-1})$ and $\mathcal{N}_{A_1/\mathcal{V}} \cong \mathcal{O}_{\mathbb{P}^{n-1}}(s)$ for a suitable integer $s > 0$. \square

Proposition 5.3. *Let \mathcal{V} be an integral normal Gorenstein projective variety of dimension $n \geq 3$ and let \mathcal{L} be an ample line bundle on \mathcal{V} . Suppose that (\diamond) holds and that $\dim \text{Irr}(\mathcal{V}) \leq 0$, where $\text{Irr}(\mathcal{V})$ is the set of irrational singularities of \mathcal{V} . Assume that each A_i is a normal Gorenstein variety such that $K_{A_i} + \tau_i \mathcal{H}_i \simeq \mathcal{O}_{A_i}$ for some integer τ_i . If $\mathcal{V} - A_k$ is a local complete intersection for some $k = 1, \dots, r$ and either $n \geq 4$, or $n = 3$ and $\mathcal{V} - F$ is a local complete intersection for some finite, possibly empty set $F \subset \mathcal{V} - \text{Irr}(\mathcal{V})$, then $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$, where Λ is an ample line bundle on \mathcal{V} , $K_{\mathcal{V}} = \rho \Lambda$, $A_i = a_i \Lambda$ and $\Lambda_{A_i} = h_i \mathcal{H}_i$ with $\tau_i = -h_i(\rho + a_i)$, where $\rho, a_i > 0$ and $h_i > 0$ are integers. In particular, for $n \geq 5$ we have $h_i = 1$ for every $i = 1, \dots, r$.*

Proof. Assume that $n \geq 4$. From case (4) of Corollary 4.1 it follows that either (a) $n \geq 5$ and $\text{Pic}(\mathcal{V}) \cong \text{Pic}(A_i) = \mathbb{Z}$ for any $i = 1, \dots, r$, or (b) $n = 4$ and $\text{Pic}(\mathcal{V})$ restricts injectively into $\text{Pic}(A_i) = \mathbb{Z}$. In both situations, we see that $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ for some ample line bundle Λ on \mathcal{V} . Moreover, in (a) we have that $\Lambda_{A_i} \simeq \mathcal{H}_i$, while in (b) we have that $\Lambda_{A_i} \simeq h_i \mathcal{H}_i$ for some positive integer h_i . By adjunction formula, we obtain that

$$\mathcal{O}_{A_i} \simeq K_{A_i} + \tau_i \mathcal{H}_i \simeq \begin{cases} (K_{\mathcal{V}} + A_i + \tau_i \Lambda)_{A_i} & \text{in case (a),} \\ (K_{\mathcal{V}} + A_i + \frac{\tau_i}{h_i} \Lambda)_{A_i} & \text{in case (b).} \end{cases}$$

i.e. $K_{\mathcal{V}} + A_i + \frac{\tau_i}{h_i} \Lambda \simeq \mathcal{O}_{\mathcal{V}}$ for some $h_i \geq 1$. If we put $K_{\mathcal{V}} = \rho \Lambda$ and $A_i = a_i \Lambda$ for some integers ρ and $a_i \geq 1$, then we see that $\tau_i = -h_i(\rho + a_i)$, with $h_i = 1$ when $n \geq 5$.

As to the case $n = 3$, from (III) of Corollary 4.3 it follows that $\text{Pic}(\mathcal{V})$ restricts injectively into $\text{Pic}(A_i) = \mathbb{Z}$. Then by arguing as above, we conclude that also in this situation $\text{Pic}(\mathcal{V}) = \mathbb{Z}\langle \Lambda \rangle$ for some ample line bundle Λ on \mathcal{V} and, with the same notation as above, $\tau_i = -h_i(\rho + a_i)$, where h_i and a_i are positive integers. \square

Remark 5.4. When $n \geq 6$, Proposition 5.3 generalizes [16, Theorem 1].

5.2 All the A_i 's have small degrees

Denote now by X an n -fold with $n \geq 3$ and by L an ample line bundle on X . In this subsection we prove the last result stated in the Introduction.

Proof of Proposition 1.3. Since $[H_i]_{A_i}^{n-1} \leq 4$, by [2, (8.10.1)] we see that $(A_i, [H_i]_{A_i})$ satisfies one of the following two conditions:

- (a) $\text{Pic}(A_i) = \mathbb{Z}\langle [H_i]_{A_i} \rangle$;
- (b) $(A_i, [H_i]_{A_i}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$.

Step (I). First of all, assume that $r = 1$. In Case (a), by Lefschetz's Theorem we see that $\text{Pic}(X) = \mathbb{Z}\langle H_1 \rangle$. Write $A_1 = a_1 H_1$ for a positive integer a_1 . Since $a_1 H_1^n = A_1 H_1^{n-1} = [H_1]_{A_1}^{n-1} \leq 4$, we deduce that $a_1 H_1^n \leq 4$. Consider the map $\varphi : X \rightarrow \mathbb{P}^N$ associated to $|H_1|$. Since $|H_1|$ is ample and spanned, the morphism φ is finite and such that

$$[H_1]^n = \deg \varphi \cdot \deg \varphi(X) \leq 4.$$

This gives the following three possibilities:

- (1) $\deg \varphi = 1$, $\deg \varphi(X) \leq 4$; (2) $\deg \varphi = 2$, $\deg \varphi(X) \leq 2$; (3) $\deg \varphi = 3, 4$ and $\deg \varphi(X) = 1$.

In (1), since $n \geq 5$ and $\text{Pic}(X) = \mathbb{Z}\langle H_1 \rangle$, by [2, (8.10.1)] (see also [6], [9], [11]) we deduce that (X, H_1) is one of the following pairs:

- $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, where $A_1 \in |\mathcal{O}_{\mathbb{P}^n}(a_1)|$ with $0 < a_1 \leq 4$;
- $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$, where $A_1 \in |\mathcal{O}_{\mathbb{Q}^n}(a_1)|$ with $0 < a_1 \leq 2$;
- $(V_d, \mathcal{O}_{\mathbb{P}^{n+1}}(1)_{V_d})$ with $A_1 \in |H_1|$, where $V_d \subset \mathbb{P}^{n+1}$ is a smooth hypersurface of degree $d = 3, 4$;
- $(W, \mathcal{O}_{\mathbb{P}^{n+2}}(1)_W)$ with $A_1 \in |H_1|$, where $W = \mathbb{Q}_1 \cap \mathbb{Q}_2 \subset \mathbb{P}^{n+2}$ is a complete intersection of two quadric hypersurfaces $\mathbb{Q}_i \subset \mathbb{P}^{n+2}$ for $i = 1, 2$.

In (2), the map φ is a double cover of either (i) \mathbb{P}^n , or (ii) $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$. In case (i), we see that $H_1 = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ and $A_1 \in |a_1 H_1|$ with $a_1 = 1, 2$. In case (ii), we get $H_1 = \varphi^* \mathcal{O}_{\mathbb{Q}^n}(1)$ and $A_1 \in |H_1|$. Finally, in (3) the morphism φ is a d -cover of \mathbb{P}^n with $d = 3, 4$, and $A_1 = H_1 = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Finally, consider Case (b). Note that $\text{Pic}(X) \neq \mathbb{Z}$. Moreover, since A_1 is ample, for any ample line bundle H on $\mathbb{P}^1 \times \mathbb{P}^3$, we have

$$\begin{aligned} 0 &= \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(2, 0) \cdot H^2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_{A_1} + 4H_{1A_1}) \cdot H^2 = \\ &= \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_X + 4H_1 + A_1)_{A_1} \cdot H^2 \geq \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0) \cdot (K_X + 5H_1)_{A_1} \cdot H^2, \end{aligned}$$

i.e. $K_X + 5H_1$ can not be ample on X . So by [10], [6] and [2, §7.2], we see that (X, H_1) is a scroll over \mathbb{P}^1 of dimension five, i.e. $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_5)$ with $1 \leq a_1 \leq \dots \leq a_5$, and H_1 is the tautological line bundle ξ of $\mathbb{P}(\mathcal{E})$. Put $\xi' = \xi - a_1 F$. Since A_1 is ample on X , write $A_1 = a\xi' + bF$ for some positive integers a and b (see [2, (3.2.4)]). Then

$$[H_1]_{A_1}^4 = (a\xi' + bF)(\xi)^4 = a(\xi - a_1 F)(\xi)^4 + b = a(a_2 + \dots + a_5) + b \geq 5,$$

but this is absurd. Thus Case (b) can not occur for $r = 1$.

Step (II). Suppose now that $r \geq 2$. First of all, let us prove the following

Claim. If $(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$, then $K_X + 4H_i$ is not nef for some $i = 1, \dots, r$.

Suppose that $K_X + 4H_k$ is nef for every $k = 1, \dots, r$. Let $\Phi_1 : A_1 \rightarrow \mathbb{P}^1$ be the nefvalue morphism of (A_1, H_{1A_1}) . Up to renaming, assume that A_2, \dots, A_s are the only components of A such that $A_1 \cap A_h \neq \emptyset$ for $h = 2, \dots, s$ with $s \leq r$. Put $h_k := [A_k]_{A_1}$ for $k = 2, \dots, r$ and note that h_t is trivial for $t = s+1, \dots, r$.

First of all, if $\Phi_1(h_k)$ is a union of points of \mathbb{P}^1 for every $k = 2, \dots, s$, then for a general fiber $F \cong \mathbb{P}^3$ of Φ_1 we have that

$$(L_{A_1})_F = ([A_1]_{A_1})_F + [h_2]_F + \dots + [h_s]_F = ([A_1]_{A_1})_F = \mathcal{O}_F(a),$$

for a suitable integer $a \geq 1$. Moreover, note that $K_{A_1} + 4H_{1A_1} \simeq 2F$.

Thus by adjunction we obtain that

$$\begin{aligned} [K_X + 4H_1]_F H_{1F}^2 &= ([K_X + 4H_1]_{A_1})_F (H_{1A_1})_F^2 = \\ &= (2F - [A_1]_{A_1})_F (H_{1A_1})_F^2 = -([A_1]_{A_1})_F (H_{1A_1})_F^2 = -\mathcal{O}_F(a) \mathcal{O}_F(1)^2 < 0, \end{aligned}$$

but this is absurd, since $K_X + 4H_1$ is nef and H_1 is ample on X .

Assume now that $\Phi_1(h_t) = \mathbb{P}^1$ for some $t = 2, \dots, s$. If $\text{Pic}(A_t) = \mathbb{Z}$, then by [3, (4.2)] we get a contradiction by taking $B = A_t$ and $A = A_1$. Thus we can assume that $(A_t, H_{tA_t}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1))$. Note that there exists a general fiber F of Φ_1 such that $F \not\subseteq h_t$ and $[h_t]_F$ is effective and not trivial. Moreover, we have that $\text{Pic}(F) \cong \mathbb{Z}$ and $F \not\subseteq A_t$, since otherwise $F \subseteq A_1 \cap A_t = h_t$. Therefore $F \cap A_t \neq \emptyset$ and $F \cap A_t \neq F$. Since

$$\dim F + \dim A_t - \dim \Phi_1(A_1) = 3 + 4 - 1 = 6 > 5 = \dim X,$$

by the same argument as in the proof of [3, (4.2)] with $B = F$ and $A = A_t$, we can conclude that $F \subseteq A_t$, but this gives a contradiction. Q.E.D.

By the above Claim, if one of the A_k , say A_1 , is such that

$$(A_1, H_{1A_1}) \cong (\mathbb{P}^1 \times \mathbb{P}^3, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 1)),$$

then $n = 5$ and $K_X + 4H_i$ is not nef for some $i = 1, \dots, r$. Since $\text{Pic}(A_1) \neq \mathbb{Z}$, by [10], [6] and [2, §7] we deduce that (X, H_i) is a \mathbb{P}^4 -bundle over a smooth curve C . Moreover, A_1 dominates C . So $C \cong \mathbb{P}^1$ and $X \cong \mathbb{P}(\mathcal{E})$ for some vector bundle \mathcal{E} of rank-5 on \mathbb{P}^1 such that $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \dots \oplus \mathcal{O}_{\mathbb{P}^1}(a_4)$ with $0 \leq a_1 \leq \dots \leq a_4$. Let ξ be the tautological line bundle of $\mathbb{P}(\mathcal{E})$. Write $A_1 = a_1\xi + b_1F$ and $H_1 = \alpha_1\xi + \beta_1F$ with $b_1 \geq 0$ and a_1, α_1, β_1 positive integers (see [2, (3.2.4)]). Furthermore, put $F_{A_1} \simeq \mathcal{O}_{A_1}(a, 0)$ and $\xi_{A_1} \simeq \mathcal{O}_{A_1}(b, c)$ with $a > 0$ and $b, c \geq 0$. By the adjunction formula we have that

$$\begin{aligned} \mathcal{O}_{A_1}(-2, -4) &\simeq K_{A_1} \simeq (K_X + A_1)_{A_1} \simeq [-5\xi + (\deg \mathcal{E} - 2)F + a_1\xi + b_1F]_{A_1} = \\ &= (a_1 - 5)\xi_{A_1} + (b_1 + \deg \mathcal{E} - 2)F_{A_1} \simeq \mathcal{O}_{A_1}(b(a_1 - 5) + a(b_1 + \deg \mathcal{E} - 2), c(a_1 - 5)). \end{aligned}$$

This gives the following two equations:

$$4 = c(5 - a_1) \tag{b}$$

$$2 = b(5 - a_1) - a(b_1 + \deg \mathcal{E} - 2). \quad (\natural)$$

Note that from (b) it follows that $1 \leq a_1 \leq 4$ and then

$$4 \geq a_1 = [A_1]_F \cdot [\xi]_F^3 = F_{A_1} \cdot \xi_{A_1}^3 = \mathcal{O}_{A_1}(a, 0) \cdot \mathcal{O}_{A_1}(b, c)^3 = ac^3.$$

Thus we deduce that $c = 1$ and $a_1 = a$. By (b) we have also that $a = a_1 = 1$ and the equation (natural) gives $4b = b_1 + \deg \mathcal{E}$. Since

$$\mathcal{O}_{A_1}(1, 1) \simeq [H_1]_{A_1} = [\alpha_1 \xi + \beta_1 F]_{A_1} = \mathcal{O}_{A_1}(\alpha_1 b + \beta_1, \alpha_1),$$

we see that $\alpha_1 = 1$ and $1 = b + \beta_1 \geq b + 1 \geq 1$, i.e. $b = 0$ and $\beta_1 = 1$. Therefore, by (natural) we get $0 = 4b = b_1 + \deg \mathcal{E} \geq \deg \mathcal{E}$, i.e. $a_1 = \dots = a_4 = 0$ and $b_1 = 0$. This shows that $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}^{\oplus 5}) \cong \mathbb{P}^1 \times \mathbb{P}^4$, $A_1 \in |\mathcal{O}_X(0, 1)|$ and $H_1 \in |\mathcal{O}_X(1, 1)|$. Consider $A_i \in |\mathcal{O}_X(d, c)|$ and $H_i \in |\mathcal{O}_X(t, s)|$ for some $i = 2, \dots, r$, where $d, c \geq 0$ and $t, s > 0$. Then we have

$$4 \geq [H_i]_{A_i}^4 = \mathcal{O}_X(d, c) \cdot \mathcal{O}_X(t, s)^4 = s^3(ds + 4ct) \geq s^3.$$

This gives $s = 1$ and $0 \leq c \leq 1$. If $c = 1$, then we get $d = 0, t = s = c = 1$ and $(A_i, [H_i]_{A_i})$ is of the same type as $(A_1, [H_1]_{A_1})$. If $c = 0$, then we see that $1 \leq d \leq 4, t \geq 1$ and $s = 1$. This shows that $A_i \in |\mathcal{O}_X(d, 0)|$ and $H_i \in |\mathcal{O}_X(t, 1)|$ with $t \geq 1$ and $1 \leq d \leq 4$, i.e. $A_i \rightarrow \mathbb{P}^4$ is a d -cover of \mathbb{P}^4 with $1 \leq d \leq 4$. Note that in this case we have

$$|\mathcal{O}_X(d, 0)| = \underbrace{|\mathcal{O}_X(1, 0)| + \dots + |\mathcal{O}_X(1, 0)|}_{d\text{-times}}.$$

Thus, since $A_i \in |\mathcal{O}_X(d, 0)|$ is irreducible and reduced, we conclude that $d = 1$.

Finally, since the above argument works independently from the choice of the component A_1 of $A \in |L|$, we can assume, without loss of generality, that every component A_i of A is such that $\text{Pic}(A_i) = \mathbb{Z}\langle [H_i]_{A_i} \rangle$. Then by Proposition 5.3 and Corollary 4.3 (III), we conclude that $\text{Pic}(X) = \mathbb{Z}\langle \Lambda \rangle$ for some ample line bundle Λ on X . Write $A_i = a_i \Lambda$ and $H_i = b_i \Lambda$ for suitable positive integers a_i and b_i . Thus we obtain that

$$a_i b_i^{n-1} \Lambda^n = (a_i \Lambda)(b_i \Lambda)^{n-1} = [H_i]_{A_i}^{n-1} \leq 4,$$

i.e. $H_1 = \dots = H_r = \Lambda$ and $a_i \Lambda^n \leq 4$. Consider the map $\varphi : X \rightarrow \mathbb{P}^N$ associated to $|\Lambda|$. Since Λ is ample and spanned, we have $\Lambda^n = \deg \varphi \cdot \deg \varphi(X) \leq 4$ and by arguing as in case $r = 1$, we obtain the statement.

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References

- [1] L. Bădescu, On ample divisors I, Nagoya Math. J. **86** (1982), 155–171.
- [2] M.C. Beltrametti, A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, Expositions in Mathematics **16**, W. de Gruyter, Berlin–New York, 1995.
- [3] K.A. Chandler, A. Howard, A.J. Sommese, Reducible hyperplane sections I, J. Math. Soc. Japan **51** (1999), 887–910.
- [4] D. Eisenbud, *Commutative Algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.
- [5] T. Fujita, *Vanishing theorems for semipositive line bundles*, In: Algebraic geometry (Tokyo/Kyoto, 1982), Springer-Verlag, Berlin, Lecture Notes in Math. **1016** (1983), 519–528.
- [6] T. Fujita, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Ser. **155**, Cambridge Univ. Press, Cambridge, 1990.
- [7] R. Hartshorne, *Ample subvarieties of algebraic varieties*, In: Notes written in collaboration with C. Musili, Lecture Notes in Math. **156**, Springer-Verlag, Berlin, 1970.
- [8] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics **52**, Springer-Verlag, New York-Heidelberg, 1977.
- [9] P. Ionescu, *Embedded projective varieties of small invariants*, In: Algebraic geometry, Bucharest 1982 (Bucharest, 1982), 142–186, Lecture Notes in Math. **1056**, Springer, Berlin, 1984.
- [10] P. Ionescu, Generalized adjunction and applications, Math. Proc. Cambridge Philos. Soc. **99** (1986), 457–472.
- [11] P. Ionescu, On manifolds of small degree, Comment. Math. Helv. **83** (2008), 927–940.
- [12] R. Lazarsfeld, *Positivity in algebraic geometry I, Classical setting: line bundles and linear series*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics **48**, Springer-Verlag, Berlin, 2004.
- [13] R. Lazarsfeld, *Positivity in algebraic geometry II, Positivity for vector bundles, and multiplier ideals*, Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics **49**, Springer-Verlag, Berlin, 2004.
- [14] A.J. Sommese, On manifolds that cannot be ample divisors, Math. Ann. **221** (1976), 55–72.
- [15] A.J. Sommese, Submanifolds of Abelian varieties, Math. Ann. **233** (1978), 229–256.
- [16] A.L. Tironi, Ample normal crossing divisors consisting of two del Pezzo manifolds, Forum Math. **22** (2010), no. 4, 667–682.

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